3

Compound Interest

Structure

3.1. Introduction. In this chapter, we discuss about certain different types of interest rates and the concept of present value and amount of a sum.

3.1.1. Objective. The objective of these contents is to provide some important results to the reader like:

(i) Different types of interest rates.

(ii) Present value.

(iii) Amount of a sum.

3.1.2. Keywords. Interest Rates, Compound Interest, Present Value.

3.2. Compound Interest.

Suppose a person takes a loan from a bank or from another person for a specified period of time. After this period, the amount he will return will be higher than the amount of loan taken. This additional amount will be paid by the borrower to the bank or second person for use of all loan given to him. This amount is called interest and the amount borrowed is called principal. Generally interest in expressed is percentage which is called rate of interest.

Interest is of two types :

1) Simple interest
2) Compound interest

If the lender is paid actual interest after every three months, six months or a year, it is called simple interest. But if this interest instead of being paid to the lender, is added to the principal and interest for next period is calculated on this new amount (principle + interest), it is called compound interest. In compound interest, the lender is paid full amount after completion of the period only once.

3.2.1. Example. Suppose Rs. 1000 is lent at 10% per annum for 2 years. Calculate simple interest and compound interest.

Solution. Simple Interest (S.I.)

$$\text{S.I. for } 1^{st} \text{ year} = \frac{1000\times10\times1}{100} = \text{Rs. } 100$$

$$\text{S.I. for } 2^{nd} \text{ year} = \frac{1000\times10\times1}{100} = \text{Rs. } 100$$

S.I. for 2 years = Rs. 100 + Rs. 100 = Rs. 200

Compund Interest (C.I.)

$$\text{C.I. for 1st year} = \frac{1000\times10\times1}{100} = \text{Rs. } 100$$

After 1 year, this interest of Rs. 100 is not given to the lender but added to his principal.

So new principal = Rs. 1000 + Rs. 100 = Rs. 1100

Now $\text{C.I. for } 2^{nd} \text{ year} = \frac{1000\times10\times1}{100} = \text{Rs. } 110$

So C.I. for 2 years = Rs. 100 + Rs. 110 = Rs. 210

3.2.2. Theorem. IF P is the principle, r % is the rate of interest per period and n is the number of periods, then

(i) Simple Interest (S.I.) = $\frac{\text{P.r.n}}{100}$ and

(ii) Compound Interest = $P\left(1+\frac{r}{100}\right)^n - P$

Proof. **(i)**

$$\text{S.I. of } 1^{st} \text{ period} = \frac{\text{P.r.1}}{100} = \frac{\text{Pr}}{100}$$

$$\text{S.I. of } 2^{nd} \text{ period} = \frac{\text{P.r.1}}{100} = \frac{\text{Pr}}{100}$$

Continuing in this way.

$$\text{S.I. of } n^{th} \text{ period} = \frac{\text{P.r.1}}{100} = \frac{\text{Pr}}{100}$$

So total S.I. for n periods = S.I. for 1st period + S.I. for 2nd period +….+ S.I. for nth period

$$= \frac{Pr}{100} + \frac{Pr}{100} + \cdots n\ times$$

$$= \frac{\mathrm{Pr}\,n}{100}$$

And amount A = P +S.I.

$$= P + \frac{\mathrm{Pr}\,n}{100}$$

$$= P\left(1 + \frac{\mathrm{r}\,n}{100}\right)$$

(ii) C.I for 1st period = $\frac{P.r.1}{100} = \frac{P.r}{100}$

Amount after 1st period = $P + \frac{Pr}{100} = P\left(1 + \frac{r}{100}\right)$

C.I. for 2nd period = $P\left(1 + \frac{r}{100}\right)\frac{r}{100}$

So amount after 2nd period = $P\left(1 + \frac{r}{100}\right) + P\left(1 + \frac{r}{100}\right)\frac{r}{100} = P\left(1 + \frac{r}{100}\right)\left[1 + \frac{r}{100}\right]$

$$= P\left(1 + \frac{r}{100}\right)^2$$

C.I. for 3rd period = $P\left(1 + \frac{r}{100}\right)^2 \frac{r}{100}$

So amount after 3rd period = $P\left(1 + \frac{r}{100}\right)^2 + P\left(1 + \frac{r}{100}\right)^2 \frac{r}{100} = P\left(1 + \frac{r}{100}\right)^2 \left[1 + \frac{r}{100}\right]$

$$= P\left(1 + \frac{r}{100}\right)^3$$

Amount after n periods = $P\left(1 + \frac{r}{100}\right)^n$

and Compound Interest

C.I. after n periods = $P\left(1 + \frac{r}{100}\right)^n - P$

$$= P\left[\left(1 + \frac{r}{100}\right)^n - 1\right]$$

Notes.

1) If n is not a whole number then it is divided into two parts – (i) a whole number part (k) and (ii) a fractional number (p) so n = k + p then

$$A = P\left(1 + \frac{r}{100}\right)^k \left(1 + \frac{Pr}{100}\right).$$

For example if n = 15 years 3 months, then n = 15 years $+\frac{1}{4}$ year and

$$A = P\left(1+\frac{r}{100}\right)^{15}\left(1+\frac{r}{4.100}\right)$$

2) Generally the unit of time period is in years. So the interest is compounded annually. In this case the above formula holds good. But if the interest is compounded monthly, quarterly or half yearly then calculations are changed as follows :

(i) Interest is compounded monthly

$$A = P\left(1+\frac{r}{12.100}\right)^{12n}$$

(ii) Interest is compound quarterly

$$A = P\left(1+\frac{r}{4.100}\right)^{4n}$$

(iii) Interest is compounded six monthly or half yearly

$$A = P\left(1+\frac{r}{2.100}\right)^{2n}$$

3) If the rate of interest (r%) changes every year i.e. r_1 in 1st year, r_2 in 2nd year,….., r_n in nth year then

$$A = \boldsymbol{P}\left(1+\frac{r_1}{100}\right)\left(1+\frac{r_2}{100}\right)\left(1+\frac{r_3}{100}\right)\cdots\left(1+\frac{r_n}{100}\right)$$

3.2.3. Example. Find the compound interest on Rs. 50000 invested at the rate of 10% for 4 years.

Solution. P = Rs. 50000, r = 10%, n = 4 years

$$A = P\left(1+\frac{r}{100}\right)^{n}$$

$$= 5000\left(1+\frac{10}{100}\right)^{4}$$

$$= 5 \times\frac{11}{10}\times\frac{11}{10}\times\frac{11}{10}\times\frac{11}{10}$$

$$= 5000 \times 14641 = Rs. 73205$$

$$C.I. = A - P = 73205 - 50000 = Rs. 23205$$

3.2.4. Example. Ram deposits Rs. 31250 in a bank at a rate of 8% per annum for 3 years. How much amount will be get after 3 years. How much his earning will change, if interest is compounded half yearly.

Solution. (i) P = Rs. 31250, r = 8%, n = 3 years

$$A = 31250\left(1+\frac{8}{100}\right)^{3}$$

$$= 31250 \times \frac{27}{25} \times \frac{27}{25} \times \frac{27}{25} = \text{Rs. } 39366$$

(ii) If the rate is compounded half yearly, then

$$A = P\left(1 + \frac{r}{2.100}\right)^{2n}$$

$$= 31250\left(1 + \frac{8}{2.100}\right)^{2\times 3}$$

$$= 31250\left(1 + \frac{1}{25}\right)^{6} = Rs. 39541.22$$

Change in earnings = $39541.22 - 39366 = Rs. 175.22$

So if the interest rate is compounded half yearly, he will earn Rs. 175.22 more.

3.2.5. Example. Find the compound interest on a sum of Rs. 100000 at the rate 12% per annum for $2\frac{1}{2}$ years when the interest is compounded (i) annually, (ii) half yearly, (iii) quarterly (iv) monthly.

Solution. $P = Rs. 100000,\ r = 12\ \%,\ n = 2\frac{1}{2}\ years.$

(i) Interest compounded annually

$$A = 100000\left(1 + \frac{12}{100}\right)^{2}\left(1 + \frac{12}{2.100}\right) = 100000 \times \left(\frac{28}{25}\right)^{2}\left(\frac{53}{50}\right)$$

$$Log\ A = Log\left[100000 \times \left(\frac{28}{25}\right)^{2}\left(\frac{53}{50}\right)\right]$$

$$= log\ 100000 + 2[log\ 28 - log\ 25] + [log\ 53 - log\ 50]$$

$$= 5 + 2[1.4471 - 1.3979] + [1.7243 - 1.6990]$$

$$= 5 + 0.0984 + .0253$$

$$= 5.1237$$

$$A = AL[5.1237] = Rs. 132953$$

So $\quad C.I. = 132953 - 100000 = Rs. 32953$

(ii) Interest is compounded half yearly

$$A = 100000\left(1 + \frac{12}{2.100}\right)^{5} = 100000\left(1 + \frac{3}{50}\right)^{5} = 100000\left(\frac{53}{50}\right)^{5}$$

$$Log\ A = log\left[100000 \times \left(\frac{53}{50}\right)^{5}\right]$$

$$= log\ 100000 + 5(log\ 53 - log\ 50) = 5 + .1265 = 5.1265$$

$$A = AL[5.1265] = Rs. 133822$$

$C.I. = 133822 - 100000 = Rs. 33822$

(iii) Interest compounded quarterly

$$A = 100000\left(1+\frac{12}{4.100}\right)^{\frac{5}{2}\times 4} = 100000\left(\frac{103}{100}\right)^{10}$$

$$Log\ A = log\left[100000 \times \left(\frac{103}{100}\right)^{10}\right]$$

$$= log\ 100000 + 10[log\ 103 - log\ 100]$$

$$= 5 + 0.1284 = 5.1284$$

$$A = AL[5.1284] = Rs. 134400$$

$$C.I. = 134400 - 100000 = Rs. 34400$$

(iv) Interest compounded monthly

$$A = 100000\left(1+\frac{12}{12.100}\right)^{\frac{5}{2}\times 12} = 100000\left(\frac{101}{100}\right)^{30}$$

$$Log\ A = log\left[100000 \times \left(\frac{101}{100}\right)^{30}\right]$$

$$= log\ 100000 + 30\ [log\ 101 - log\ 100]$$

$$= 5 + 30\ [0.00432]$$

So $A = AL[5.1296] = Rs. 134785$

$C.I. = 134785 - 100000 = Rs. 34785$

3.2.6. Example. At what rate % will Rs. 32768 yield Rs. 26281 as compound interest in 5 years.

Solution. P = Rs. 32768

A = P + C.I.

= 32768 + 26281 = Rs. 59049

n = 5 years

Now

$$A = P\left(1+\frac{r}{100}\right)^n$$

or

$$59049 = 32768\left(1+\frac{r}{100}\right)^5$$

or

$$\frac{59049}{32768} = \left(1+\frac{r}{100}\right)^5$$

or

$$\left(\frac{9}{8}\right)^5 = \left(1+\frac{r}{100}\right)^5$$

Therefore

$$1+\frac{r}{100} = \frac{9}{8}$$

$$r = \frac{100}{8} = 12.5\%$$

3.2.7. Example. At what rate % will a principal double itself in 6 years.

Solution. $A = P\left(1+\frac{r}{100}\right)^n$

$$2P = P\left(1+\frac{r}{100}\right)^6$$

or

$$\left(1+\frac{r}{100}\right)^6 = 2$$

Let $1+\frac{r}{100} = x$, then $x^6 = 2$. Taking logarithms of both sides

$6\, log\, x = log\, 2 = 0.3010$

or $log\, x = 0.0502$

Therefore $x = AL[0.0502] = 1.1225$

So now $1+\frac{r}{100} = 1.1225$

or $r = 12.25\,\%$

3.2.8. Example. In how many years will Rs. 30000 becomes Rs. 43923 at 10% rate of interest.

Solution. $A = P\left(1+\frac{r}{100}\right)^n$

$$43923 = 30000\left(1+\frac{10}{100}\right)^n$$

or

$$\frac{43923}{30000} = \left(\frac{11}{10}\right)^n$$

or

$$\left(\frac{11}{10}\right)^4 = \left(\frac{11}{10}\right)^n$$

Therefore, n=4.

So in 4 years Rs. 30000 will become Rs. 43923 at 10% rate of interest.

3.2.9. Example. Sita invested equal amounts are at 8% simple interest and the other at 8% compound interest. If the latter earns Rs. 3466.40 more as interest after 5 years, find the total amount invested.

Solution. Let amount invested in each = P

So S.I. on P for 5 years at 8% = $\frac{P\times8\times5}{100} = \frac{2}{5}P$

and C.I. on P for 5 years at 8% $= \mathrm{P}\left(1+\frac{8}{100}\right)^5 - \mathrm{P}$

$$= \mathrm{P}\left[\left(\frac{27}{25}\right)^5 - 1\right]$$

$$\text{Difference} = \mathrm{P}\left[\left(\frac{27}{25}\right)^5 - 1\right] - \frac{2}{5}\mathrm{P}$$

$$= \mathrm{P}\left[\left(\frac{27}{25}\right)^5 - 1 - \frac{2}{5}\right]$$

$$= \mathrm{P}\left[\left(\frac{27}{25}\right)^5 - \frac{7}{5}\right] = \mathrm{P}\left[\frac{14348907}{9765625} - \frac{7}{5}\right]$$

$$= \frac{677032}{9765625}P$$

So now $\frac{677032}{9765625}P = 3466.40$

or $P = \frac{3466.40\times9765625}{677032} = Rs. 50000$

So total amount invested = 50000 + 50000

= Rs. 100000

3.2.10. Example. A sum of money invested at C.I. becomes Rs. 28231.63 after 4 years and Rs. 3542.00 after 6 years. Find the principal and the rate of interest.

Solution. Let principal be P and rate of interest be r.

So $28231.63 = P\left(1+\frac{r}{100}\right)^4$ … (i)

And $33542.00 = P\left(1+\frac{r}{100}\right)^6$ … (ii)

Dividing (ii) by (i)

$$\frac{33542}{28231.63} = \left(1+\frac{r}{100}\right)^2$$

Put $1+\frac{r}{100} = x$, therefore

$$\frac{33542}{28231.63} = x^2$$

Taking logarithms of both sides

$$log\ 33542 - log\ 28231.63 = 2\ log\ x$$

$$4.52559 - 4.45073 = 2\ log\ x$$

or $$2 log\ x = .07486$$

$$log\ x = 0.03743$$

or $$x = AL[0.03743] = 1.09$$

Therefore $1 + \frac{r}{100} = 1.09$

$$\frac{r}{100} = 1.09 - 1 = 0.09$$

or $r = 9\%$

Now substituting this value in equation (i)

$$28231.63 = P\left(1 + \frac{9}{100}\right)^4$$

or $$P = 28231.63\left(\frac{100}{109}\right)^4 = Rs. 20000$$

3.2.11. Example. The difference between S.I. and C.I. on a certain sum of money for 3 years at $8\frac{1}{2}\%$ rate of interest is Rs. 3566.26. Find the sum.

Solution. Let principal = Rs. P

$$S.I. = \frac{x \times 3 \times 17}{2 \times 100} = \frac{51}{100}P$$

$$C.I. = P\left(1 + \frac{17}{2 \times 100}\right)^3 - P$$

$$= P\left[\left(\frac{217}{200}\right)^3 - 1\right] = \frac{2218313}{8000000}P$$

Therefore $\frac{2218313}{8000000}P - \frac{51}{200}P = 3566.26$

or $\frac{2218313P - 2040000P}{8000000} = 3566.26$

or $\frac{178313}{8000000}P = 3566.26$

$$P = \frac{3566.26 \times 8000000}{178313}$$

$$= \frac{356626}{100} \times \frac{8000000}{178313} = Rs.160000$$

3.2.12. Example. A person invests a part of Rs. 221000 at 10% C.I. for 5 years and remaining part for three years at the same rate. At time of maturity amount of both the investments is same. Find the sum deposited in each option.

Solution. Let principal in first option = P, r = 10% and n = 5 years

$$A = P\left(1 + \frac{10}{100}\right)^5 = P\left(\frac{11}{10}\right)^5$$

Sum invested in 2nd option = $(221000 - P)$

$$A = (221000 - P)\left(1 + \frac{10}{100}\right)^3 = (22100 - P)\left(\frac{11}{10}\right)^3$$

Now $\qquad P\left(\frac{11}{10}\right)^5 = (22100 - P)\left(\frac{11}{10}\right)^3$

or $\qquad \left(\frac{11}{10}\right)^2 = (22100 - P)$

or $\qquad 121\,P = 22100000 - 100P$

or $\qquad 221\,P = 22100000$

or $\qquad P = \frac{22100000}{221} = Rs\ 100000$

So the sum invested in first option is Rs. 100000 and the sum invested in 2nd option is $(221000 - 100000)$ Rs. 121000

3.3. Continuos Compounding of Interest.

If the interest rate is compounded continuously, such that compounding frequency (δ) is infinitely large then

$$A = \lim_{\delta\to\infty} P\left[1 + \frac{r}{\delta.100}\right]^{n.\delta}$$

$$A = \lim_{\delta\to\infty} P\left[1 + \frac{r}{100\delta}\right]^{\left(\frac{100\delta}{r}\right)\left(\frac{nr}{100}\right)}$$

$$P = \left[\lim_{\frac{100\delta}{r}\to\infty}\left(1 + \frac{r}{100\delta}\right)^{\frac{100\delta}{r}}\right]^{\frac{nr}{100}}$$

$$= Pr^{\left(\frac{nr}{100}\right)} \quad \left[since\ \lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^m = e\right]$$

3.3.1. Example. Rs 8000 are invested at 6% per annum. Find the amount after 5 years if interest is compounded continuously.

Solution. $A = Pe^{\left(\frac{nr}{100}\right)}$

Now P = Rs. 8000, n = 3 years and r = 6%

$$A = Rs.(2.71825)^{\frac{3\times6}{100}} = 8000 \times 1.197 = Rs. 9576.$$

3.3.2. Example. At what rate %, a sum will be doubled in 5 years if interest is compounded continuously.

Solution. $A = Pe^{\left(\frac{nr}{100}\right)}$

So $2P = P.e^{\left(\frac{5r}{100}\right)}$

or $2 = e^{\frac{r}{20}}$

Taking logarithms of both sides

$$\log 2 = \frac{r}{20}\log e$$

$$0.3010 = \frac{r}{20} \times 0.4343$$

or $$r = \frac{20\times.3010}{.4343} = 13.86\%$$

3.4. Effective Rate of Interest.

As we have seen in the example 3 that we get higher yields, if instead of annual compounding, interest is compounded monthly, quarterly or half yearly. So at the same rate of interest, we get higher interest as a result of increased compounding interest. Similarly if we want same interest in a given period, the effective rates will be higher if interest is compounded monthly, quarterly or half yearly instead of annually.

3.4.1. Example. A company offers 13% interest rate per annum on its debentures. What are the effective rates if interest is compounded (i) half yearly. (ii) quarterly (iii) monthly and (iv) continuously.

Solution. Let principal = Rs 100

Time = 1 year

Therefore , C.I. at 13% = $\frac{100\times13\times1}{100} = Rs. 13.$

(i) Interest compounded half yearly

$$A = P\left(1 + \frac{r}{2\times100}\right)^{2n}$$

$$= 100\left(1 + \frac{13}{2\times100}\right)^{2\times1}$$

$$= 100 \times \frac{213}{200} \times \frac{213}{200} = 113.42$$

So effective rate of interest = $113.42 - 100 = 13.42\%$

(ii) Interest is compounded quarterly

$$A = P\left(1 + \frac{13}{4 \times 100}\right)^4$$

$$= 100 \times \left(\frac{413}{400}\right)^4 = 100 \times \frac{2.91 \times 10^{10}}{2.56 \times 10^{10}} = Rs.\,113.67$$

So effective rate of interest = $113.67 - 100 = 13.67\%$

(iii) Interest is compounded monthly

$$A = P\left(1 + \frac{13}{12 \times 100}\right)^{12}$$

$$= 100 \times \left(\frac{1213}{1200}\right)^{12} = 100 \times \frac{10.147 \times 10^{10}}{8.916 \times 10^{10}} = Rs.\,113.81$$

So effective rate or interest = $113.81 - 100 = 13.81\%$

(iv) Interest rate compounded continuously

$$A = Pe^{\left(\frac{nr}{100}\right)}$$

$$= 100 \times e^{\left(\frac{13 \times 1}{100}\right)}$$

$$= 100 \times (2.71828)^{.13}$$

$$= 100 \times 1.1388 = 113.88$$

So effective interest rate = $113.88 - 100 = 13.88\%$

So we can see that as frequency of compounding increases, effective interest rate also goes on increasing.

3.5. Check Your Progress.

1. Find the amount after 3 years if Rs. 16000 is invested at a rate of 10% per annum.
2. Find the compound interest earned on Rs. 5000 at a rate of 8% p.a. for 5 years.
3. Find the amount and compound interest on a sum of Rs. 80000 for $2\frac{1}{2}$ years at a rate of 6.5% p.a.
4. Find the difference in compound interest if interest is compounded (i) annually and (ii) half yearly on a sum of Rs. 20000 for 3 years at a rate of 6% p.a.
5. Find compound interest on Rs. 5000 at 8% p.a. compounded quarterly for nine months.
6. At what rate percent when annum will a sum double itself in 5 years.
7. At what rate percent per annum will Rs 20000 become Rs. 30000 in 3 years if the interest is compounded (i) half yearly and (ii) quarterly.
8. A person borrows certain amount of money at 3 % per annum simple interest and invests it at 5%

p.a. compound interest. After three years, he makes a profit of Rs 5410. Find the amount borrowed.

9. In how much time will a sum be doubled if the rate of interest is 10% per annum.
10. A certain sum of money becomes Rs. 5995.08 after 3 years at 6% p.a. find the principal.
11. The compound interest on a certain sum for 4 years at 8% rate is Rs. 404.89 more than simple interest on the same sum at the same rate and for the same time. Find the principal.
12. A sum of money amounts to Rs. 8988.8 in 2 years and to Rs. 10099.82 in 4 years at compound interest. Find the principal and the rate of interest.
13. Difference between C.I. and C.I. on a certain sum of money for 2 years at 5% p.a. is Rs 10. Find the sum.
14. A sum of Rs. 16896 is to be invested in two schemes one for 3 years and the other for 2 years. Rate of interest in both the schemes is 6.25% p.a. If the amount received at the maturity of the two schemes is same, find the sum invested in each scheme.
15. In how many years will a money treble itself at 8% if the interest is compounded continuously?
16. A company offers 12% rate of interest p.a. on its deposits. What is the effective rate of interest if it is compounded (i) six monthly (ii) quarterly (iii) monthly and (iv) continuously.
17. Which is better investment 8% compounded half yearly or 7.5% compounded quarterly.

Answers

1. Rs 21296	2. Rs. 2346.64	3. Rs. 93686.98 and Rs. 13686.98	
4.Rs. 60.73	5. Rs.307	6. 14.87 %	7. (i) 14%, (ii) 13.76%
8. Rs. 15912	9. 7.27 years	10. Rs. 5034	11. Rs. 10000
12. Rs.8000 & 6%	13. Rs. 4000	14. Rs. 8192 and Rs. 8704	
15. 8.53 years	16. (i) 12.36% (ii) 12.55% (iii) 12.68% and (iv) 12.75%		
17. Ist option			

3.6. Summary. In this chapter, we discussed compound interest, continuous compounding of interest, effective rate of interest and some examples related to these topics which help in understanding.

Books Suggested.

1. Allen, B.G.D, Basic Mathematics, Mcmillan, New Delhi.
2. Volra, N. D., Quantitative Techniques in Management, Tata McGraw Hill, New Delhi.
3. Kapoor, V.K., Business Mathematics, Sultan chand and sons, Delhi.